Magnitude Distance

A Geometric Measure of Dataset Similarity

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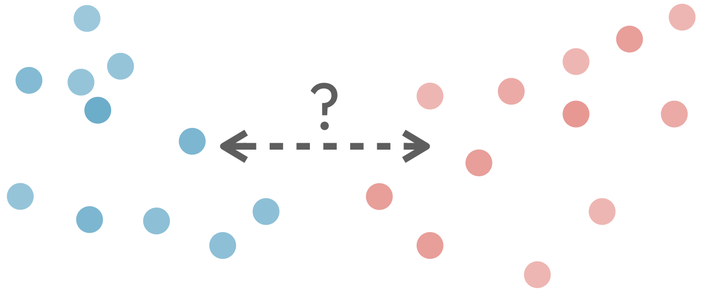
School of Informatics, University of Edinburgh

The Limits of Classical Distances

Classical Distances Lose Discriminability in High Dimensions.

Comparing datasets is fundamental, from hypothesis testing to training generative models.



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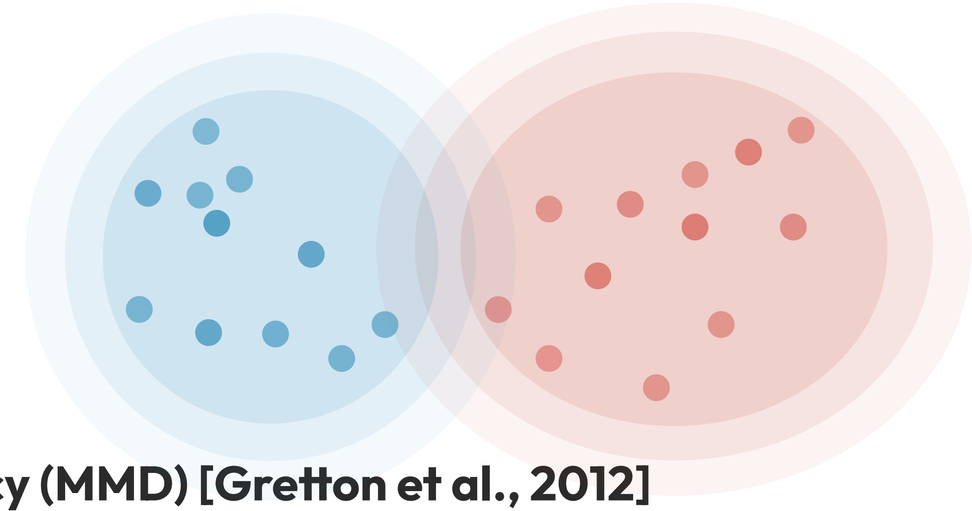

✗ **Maximum Mean Discrepancy (MMD) [Gretton et al., 2012]**

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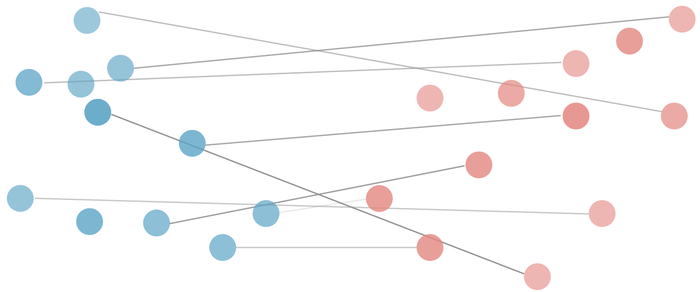

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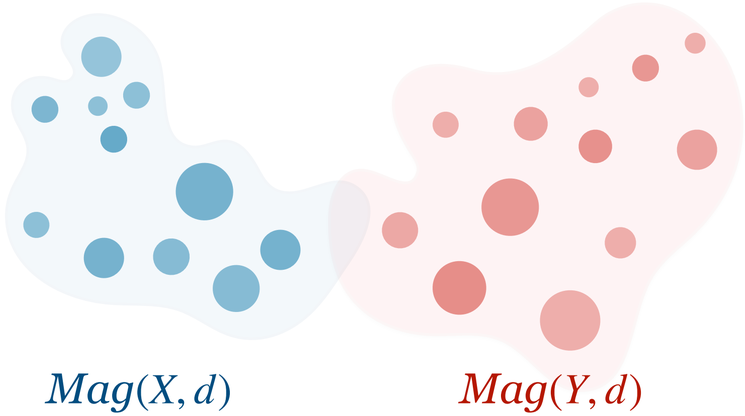
✗ **Wasserstein distance [Givens & Shortt, 1984]**

- Pairwise distances concentrate in high dimensions.
- Unbounded sensitivity to outliers: a single outlier can increase the distance arbitrarily.

The Key Concept

Magnitude of a Metric Space [Leinster (2013)]

Definition: Magnitude of a finite metric space (X, d) is defined as $Mag(X, d) = \mathbf{1}^T \zeta_X^{-1} \mathbf{1}$ with the similarity matrix $\zeta_X(x_i, x_j) := \exp(-d(x_i, x_j))$.



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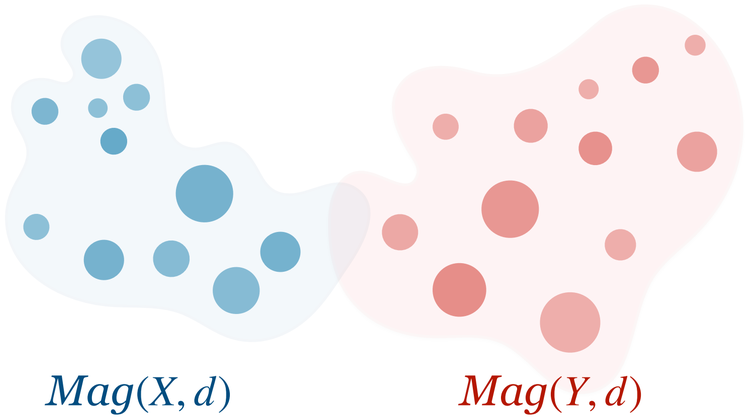

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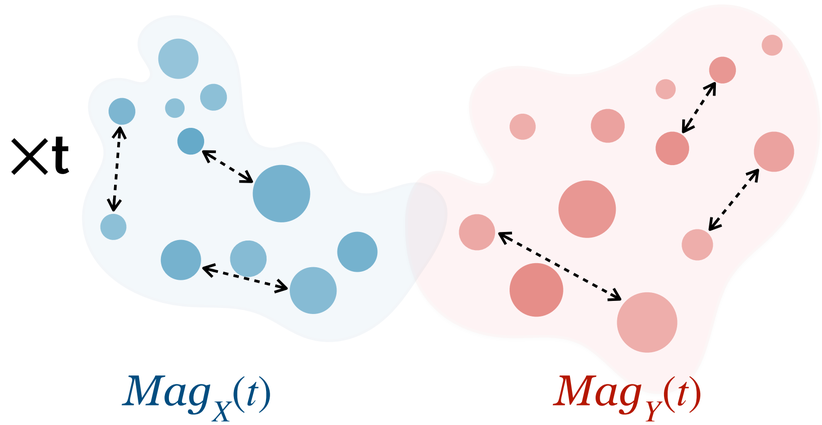

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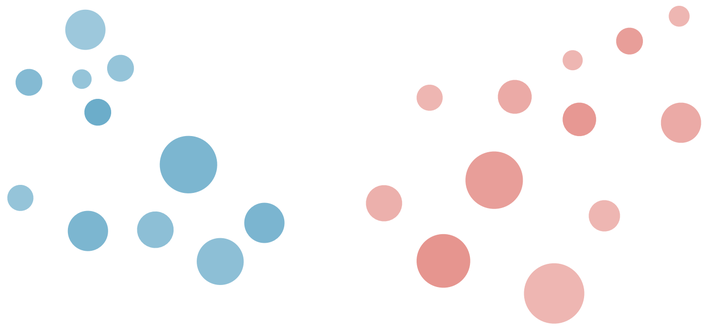

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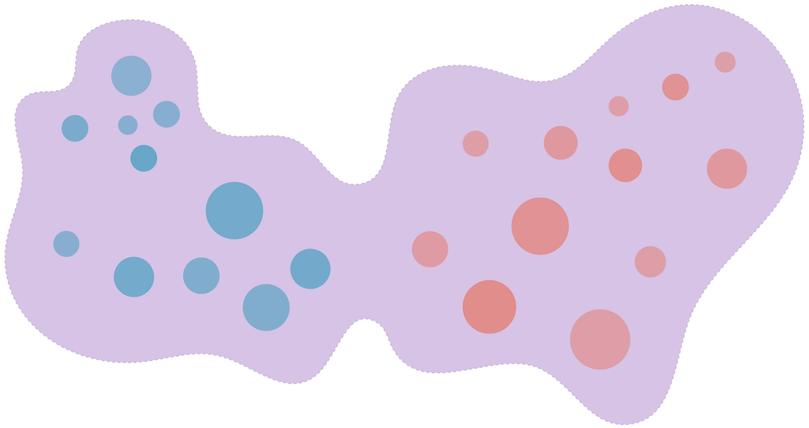

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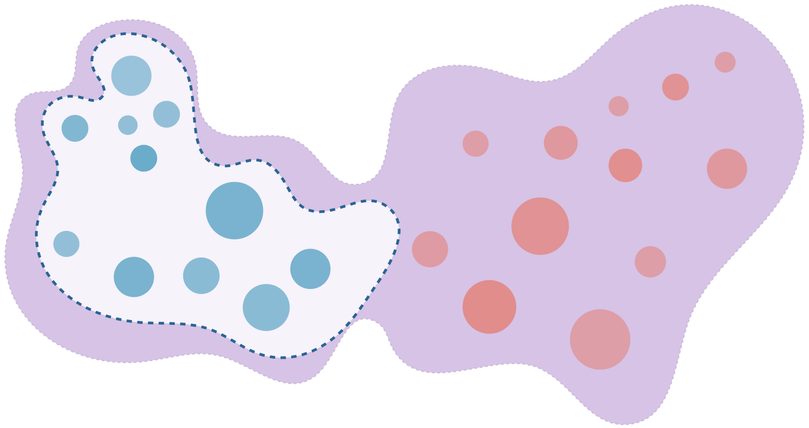

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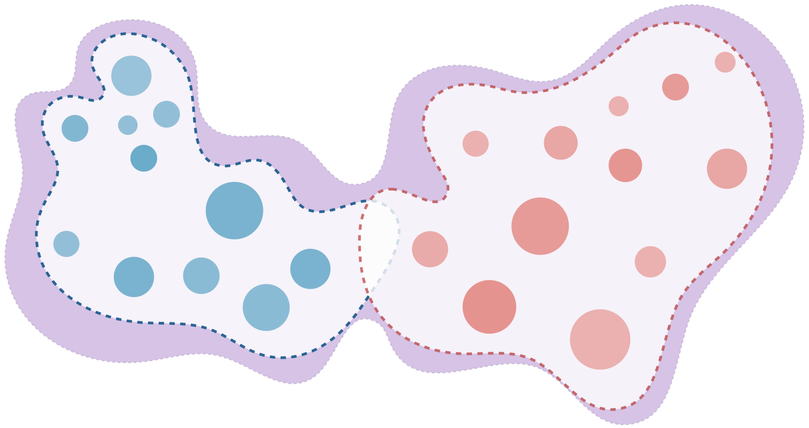

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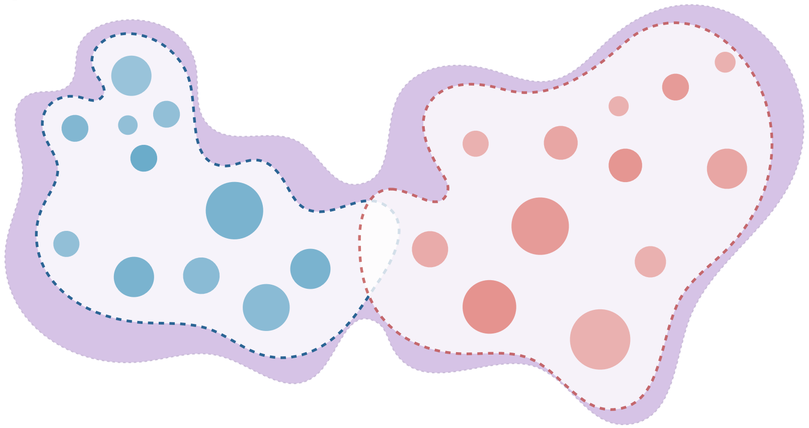

Metric axioms:

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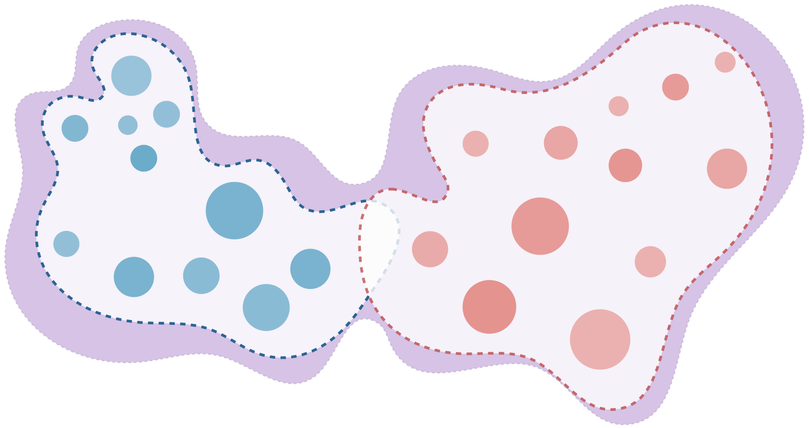

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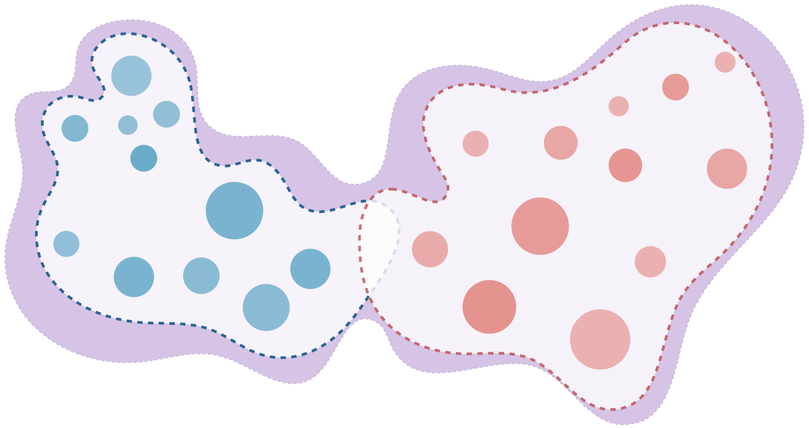

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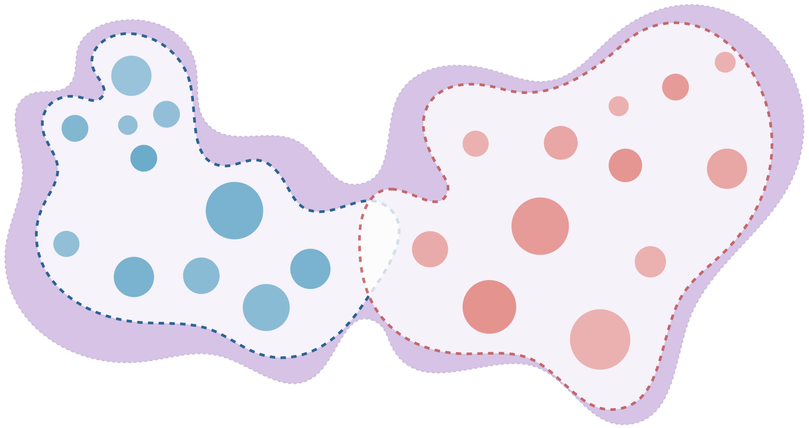

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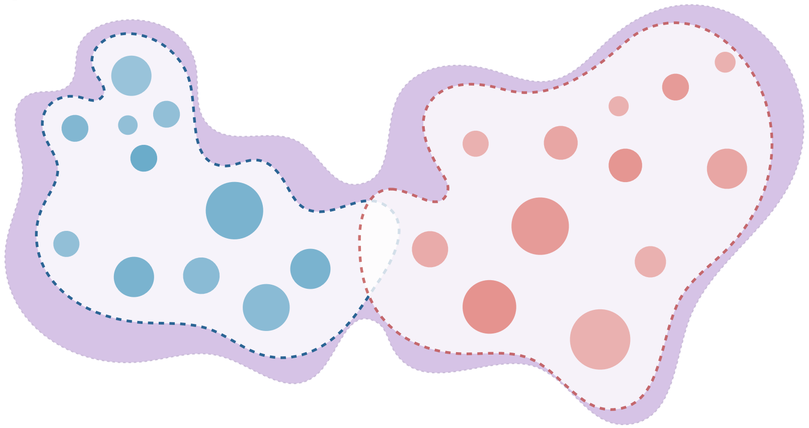

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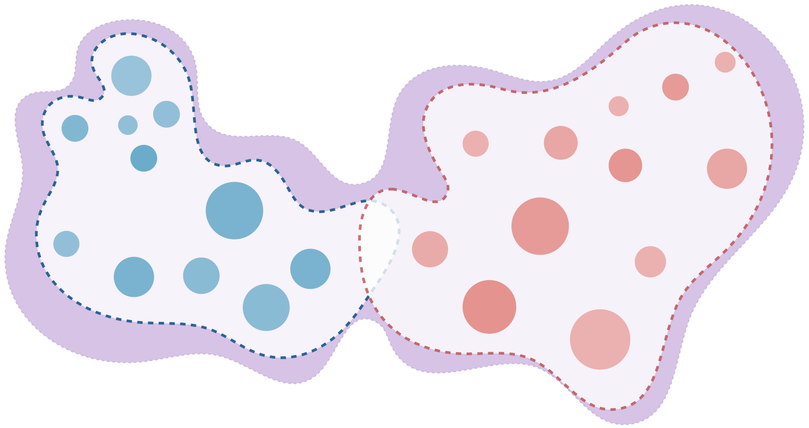

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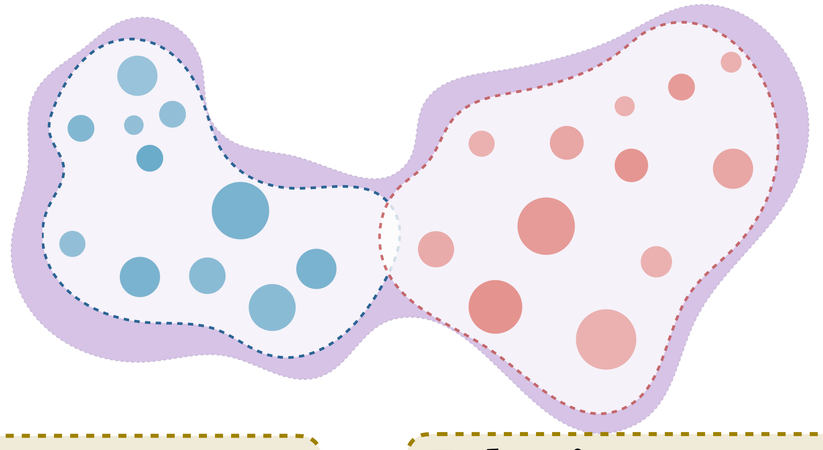

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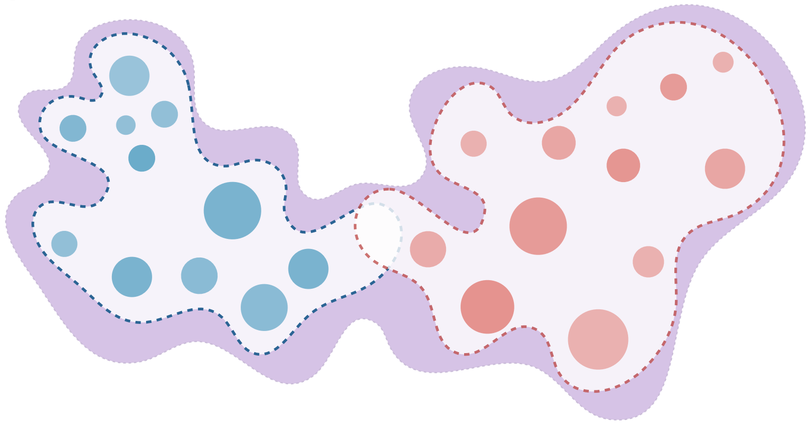

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Theoretical Guarantees

Comparison with Classical Distances

High Dimension Discriminability

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Outlier Robustness

- Wasserstein distance changes unboundedly with one outlier addition.
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Definition: For some $k \in \mathbb{N}$ and a list called t-steps $\{e_1, \ldots, e_k\}$, at epochs e_i we add $\hat{d}^{t_i}_{Mag}(\mathscr{D}_r, \mathscr{D}_g)$ to the loss, respectively, with the magnitude scaling parameters t_i where $t_1 \leq \ldots \leq t_k$.

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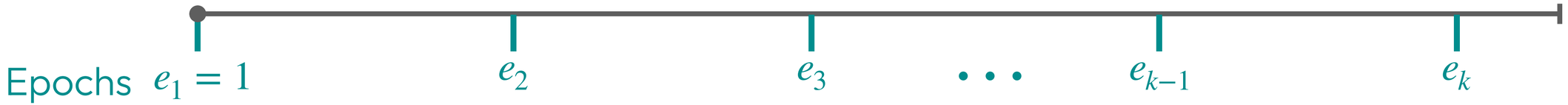

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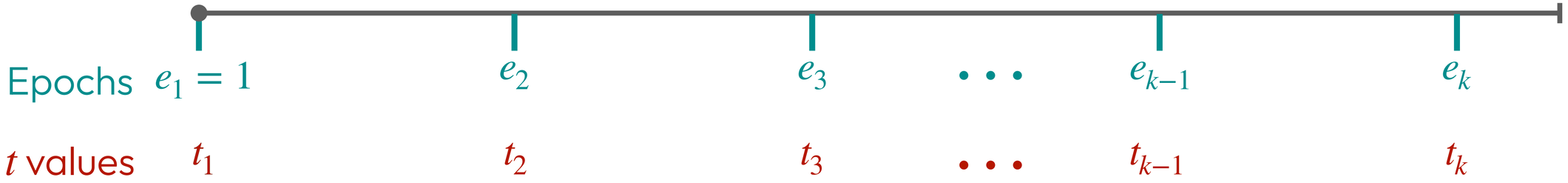

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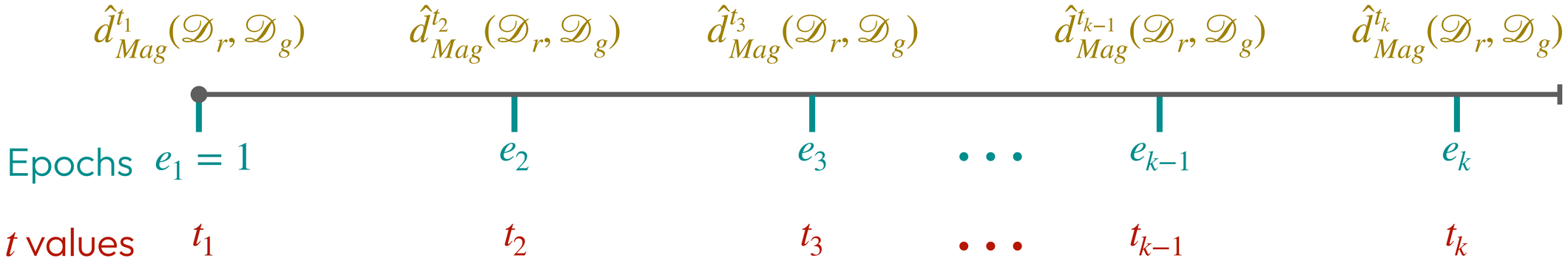
real data

generated data



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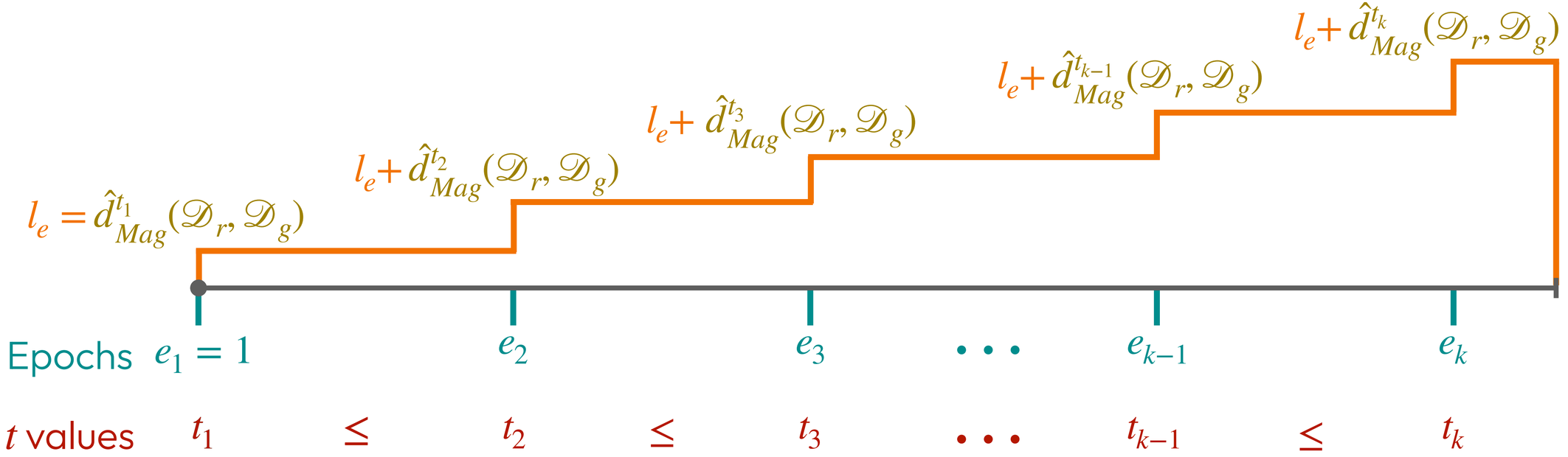
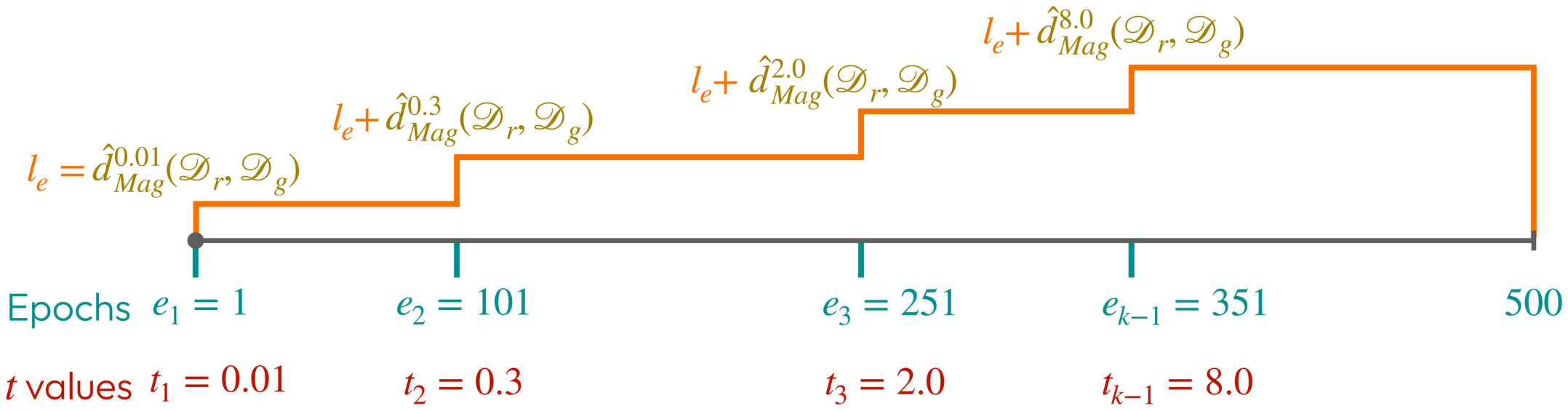


Magnitude Generative Network (MagGN)

Performance

MNIST Experiment



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$l_e = \hat{d}^{0.01}_{Mag}(\mathscr{D}_r, \mathscr{D}_g)$

$l_e + \hat{d}^{0.3}_{Mag}(\mathscr{D}_r, \mathscr{D}_g)$

$l_e + \hat{d}^{2.0}_{Mag}(\mathscr{D}_r, \mathscr{D}_g)$

$l_e + \hat{d}^{8.0}_{Mag}(\mathscr{D}_r, \mathscr{D}_g)$

Epochs $e_1 = 1$ $e_2 = 101$ $e_3 = 251$ $e_{k-1} = 351$ 500

t values $t_1 = 0.01$ $t_2 = 0.3$ $t_3 = 2.0$ $t_{k-1} = 8.0$

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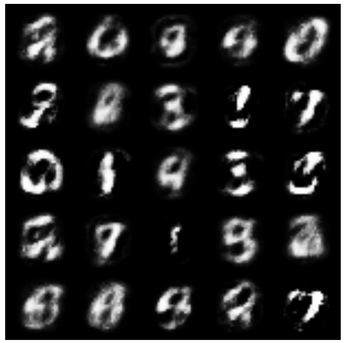
WGAN



WGAN-GP



MagGN



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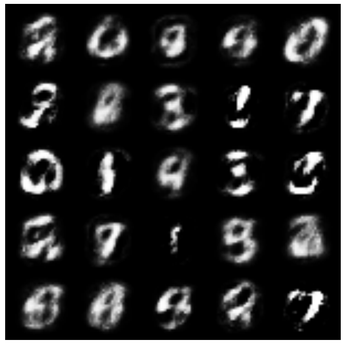

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Observation:
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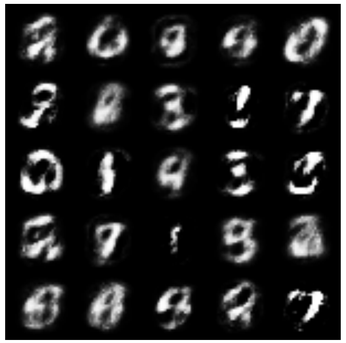

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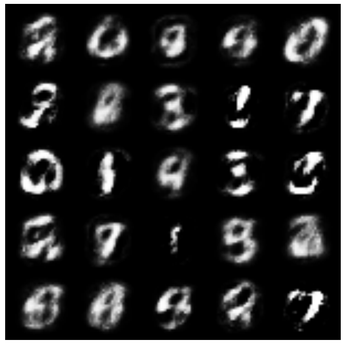

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x12 Faster!

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Summary

A new Measure for Comparing Datasets

- Magnitude distance is **tunable** via scaling parameter: from global structure to fine-grained local detail.
- Remains **discriminative in high dimensions**, where classical distances collapse.
- As a proof of concept, **MagGN** achieves >12× training speedup with no loss in visual quality.
- We expect magnitude distance to be used in various machine learning contexts as a **fundamental tool**.